### Optimization of hierarchical structures of information flow

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#### Abstract

The efficiency of a large hierarchical organisation is simulated on Barabási-Albert networks, when each needed link leads to a loss of information. The optimum is found at a finite network size, corresponding to about five hierarchical layers, provided a cost for building the network is included in our optimization.

## 1 Introduction

In a hospital, the information on how to treat a patient has to travel from the experts to the nurses, and errors may occur in this process. They can be avoided if only the leading expert deals with the patients, but then only few patients can be treated. In the opposite extreme one builds in many layers of hierarchy where the information has to be given from each layer to the larger layer below it until the nurses get the information. For a large number of layers, growing in size from top to bottom, one then treats badly a huge number of patients. What is the optimum number of layers, or more generally, the optimum organisational structure, for the hospital?

Analogous problems occur elsewhere in society [1]. Leaders (government, university president, company CEO) think that they know best, but we minor subjects not always fully appreciate their wisdom since something got lost in translation from one level to the lower level.

We assume the loss of information from one layer to the next to be a fixed fraction x either of the initial information (linear decay) or of the current

information (exponential decay). If the top layer has index L = 0, then at layer L of the original information only the fraction  $\exp(-xL)$  or 1 - xL arrives, for exponential or linear decay, respectively.

In a directed square lattice such information flow was already treated in an old Citation Classic [2]. In the next section, we deal with it on a Cayley tree analytically, while in section 3 we simulate it on directed Barabási-Albert networks [3].

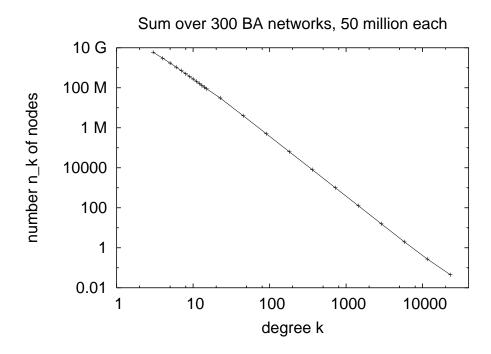


Figure 1: Number  $n_k$  of nodes having k neighbours each, summed over 300 Barabási-Albert networks with m = 3 and 50 million nodes each.

# 2 Cayley tree

Imagine there is one omniscient expert on top in layer 0, who talks to b subordinates in layer 1, each of which again talks to b different subordinates in layer 2, etc. With L layers below the top we have in total

$$N = (b^{L+1} - 1)/(b - 1) \tag{1}$$

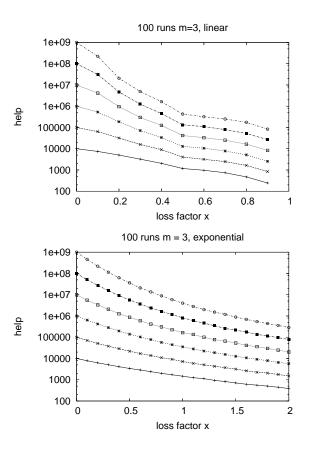


Figure 2: Help summed over 100 scale-free networks of  $N=10^2$ ,  $10^3$ ,  $10^4$ ,  $10^5$ ,  $10^6$ ,  $10^7$  nodes each, with m=3, versus loss fraction x and linear (top) or exponential (bottom) information decay.

people, and the bottom layer has a distance of L from the top (as measured by the number of connecting links.) This Cayley tree or Bethe lattice is well known to be analytically solvable in many applications. We measure the help H (or profit, or utility function) by the total amount of information arriving in the bottom layer. This help is

$$H = (1 - xL)b^L$$
 linear, for  $L < 1/x$  (2a)

or

$$H = e^{-xL}b^L \quad \text{exponential} \tag{2b}$$

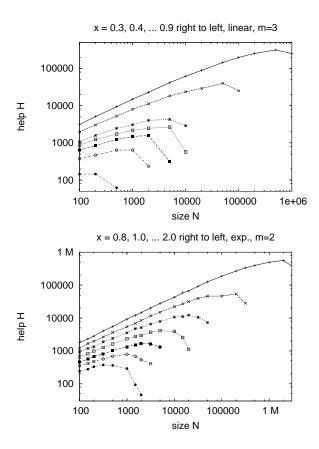


Figure 3: Help minus size/100, summed over 100 samples, for linear and exponential decay.

in our two choices. For the linear decay, the help is maximal for L near  $x^{-1} - 1/\ln b$ ; for the exponential decay one has either  $H \to \infty$  (percolation) or  $H \to 0$  (no percolation) depending on whether x is smaller or larger than  $\ln b$ .

One may also look for a maximum of H under the condition that the total number N of people, eq.(1), is constant. Then it is best to take L = 1, b = N - 1, since then everybody is close to the truth,

We doubt that these simple models and results are suitable ways to organise hospitals or other social organisations.

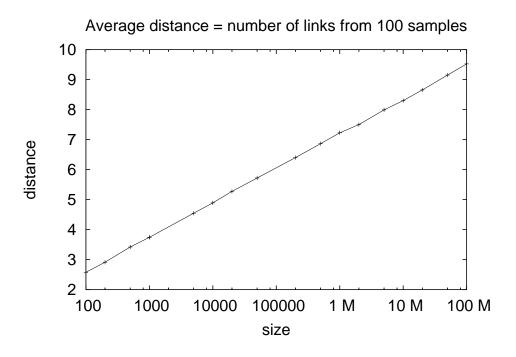


Figure 4: Number of layers, defined as the average distance  $\ell$  from the core, versus number N of nodes in 100 Barabási-Albert networks with m=2.

## 3 Scale-free networks

Some but not all social relations may be better approximated by scale-free networks [3, 4, 5]. We use the Barabási-Albert version, where the number  $n_k$  of network nodes having k neighbours decays as  $1/k^3$ , Fig. 1. We start with m fully connected nodes, and then each node which is newly added to the network selects m existing nodes as "bosses", with a probability proportional to the number of neighbours the boss has at that time: preferential attachment.

While our information flow is directed, we do not need to take this direction into account since no neighbour relations need to be stored. Instead, for each newly added node we determine the shortest distance L from the initial core of m nodes; this core has L = 0. Since now we have no longer a clear separation into layers, we assume that everybody except the core members helps the patients, with the fraction  $f_i = \exp(-xL_i)$  (exponential decay) or

 $f_i = 1 - xL_i$  (linear decay). Then

$$H = \sum_{i=m+1}^{N} f_i \tag{3}$$

is the total help.

Fig.2 shows how the resulting help decreases with increasing loss fraction x. It confirms, not surprisingly, that the help decreases with increasing information loss x and increases with increasing network size N. (For fixed  $N=10^5$  and  $2 \le m \le 7$  the help increases slightly with increasing m.) It is more realistic to include a cost associated with the network; thus we look at the modified help

$$H' = H - \lambda N, \quad \lambda = 0.01 \quad . \tag{4}$$

Fig.3 shows that this function has a maximum in the size range of interest, except for small x which would require even bigger networks. The logarithmic horizontal axis of Fig.3 corresponds to a linear axis in the layer number  $< \ell >$ , since the latter varies as log N, Fig.4. Also in the Cayley tree one can find such a maximum help if one subtracts 0.01N from H, eqs.(1,2b)

## 4 Conclusion

For scale-free Barabási-Albert networks and Cayley trees, we found a maximum in the desired help function at network sizes, which correspond to numbers of layers larger than two and smaller than ten, a reasonable result.

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## References

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